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INEQUALITIES of ORDERS on LOCAL ANALYTIC ALGEBRAS

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1. Orders and generic rank

Let X be an analytic space over $k=\mathbf{R}$ or $\mathbf{C}$ and $(O,m) =(O_{X,\xi},m_{\xi})$ the local ring at $\xi\in X$. O can be expressed as $O=k\{x\}/I$ for an algebra $k\{x\}$ of convergent power series in $x=(x_1,\dots,x_n)$ and an ideal $I\subset k\{x\}$. We define three kinds of orders for $f\in O$ as follows.

algebraic order: $\nu(f):=\sup\{p: f\in m^p\}$

reduced order ([L-T]): $\bar{\nu}(f):=\lim_{k\to\infty}\nu(f^k)/k$

analytic order along $A\subset|X|$: $\mu_{A,\xi}(f):=\sup\{p: \exists\alpha>0, \exists$ nbd. U of ξ, $\exists$ representative $\tilde{f}$ of f over U such that $|f(x)|\leqq\alpha|x-\xi|^p$ for $x\in A\cap U\}$

We know the following inequalities.

$\exists N\in\mathbf{N}$: $\nu(f)\leqq\bar{\nu}(f)\leqq\nu(f)+N$ $(f\in O)$ ([L-T])

if $A\supset B$, $\mu_{A,\xi}(f)\leqq\mu_{B,\xi}(f)$ $(f\in O)$

Example 1. If $O=\mathbf{R}\{x,y\}$ and if $A=\{(x,y): |y|\leqq|x|^p\}$ $(p\geqq1)$, then $\nu(y)=1$ and $\mu_{A,0}(y)=p$.

Example 2. If $O=\mathbf{R}\{x,y\}/(y^2-x^3)$, then $\nu(y)=1$ and $\bar{\nu}(y)=3/2$.

Let $\Phi: Y\longrightarrow X$ be an anlytic map such that $\Phi(\eta)=\xi$. We define the generic rank $\mathrm{grnk}_\eta\Phi$ of Φ at η as follows.

$\mathrm{grnk}_\eta\Phi:=\varepsilon\cdot\inf\{$the topological dimension of $\Phi(U)$: U is a nbd. of $\eta\}$ ($\varepsilon=1$ if $k=\mathbf{R}$ and $\varepsilon=1/2$ if $k=\mathbf{C}$).

2. The theorem of Lejeune and Teissier

The following is the most basic result on orders on complex analytic algebras.

A. Lemma ([L-T], the original form is more general). Let X be a complex space reduced at ξ. Then, for $f \in O$, the following conditions are equivalent.

(r) $\bar{\nu}(f) \geqq p$.

(l) $\mu_{|X|,\xi}(f) \geqq p$.

(i) $\exists s \in \mathbb{N}, \exists \sigma_i \in m^{pi}$: $f^s - \sigma_1 f^{s-1} + \sigma_2 f^{s-2} - \ldots \pm \sigma_s = 0$.

(e) For $\forall$ (or $\exists$) proper surjective analytic map $\Pi: Y \longrightarrow X$ such that Y is normal and mO_Y is invertible, there exists a representative $\tilde{f}$ of f over U such that $\tilde{f} \in m^p O_Y(\Pi^{-1}(U))$.

(c) For any analytic map $\Phi: D \longrightarrow X$ with $\Phi(0)=\xi$, we have $\nu_0(f \circ \Phi) \geqq p \cdot \inf\{\nu_0(g \circ \Phi): g \in m\}$ (D is the unit open disc in $\mathbf{C}$).

3. The main theorem

The following inequalities are well-known.

(1) $\nu(fg) \geqq \nu(f) + \nu(g)$ $(f, g \in O)$.

(2) $\nu_\eta(f \circ \Phi) \geqq \nu_\xi(f)$ $(f \in O)$ for any analytic map $\Phi: (Y,\eta) \longrightarrow (X,\xi)$.

(3) $\mu_{A,\xi}(f) \geqq \nu(f)$ $(f \in O)$ for any $A \subset |X|$.

B. Theorem. Let X be a complex space reduced and irreducible at ξ (i.e. O is an integral domain) or a real analytic space whose complexification is reduced and irreducible at ξ (i.e. $O \otimes_{\mathbb{R}} \mathbf{C}$ is an integral domain). Then we have the following.

(1) $\exists a_1 \geqq 1, \exists b_1 \geqq 0$: $\nu(fg) \leqq a_1(\nu(f) + \nu(g)) + b_1$ $(f, g \in O)$.

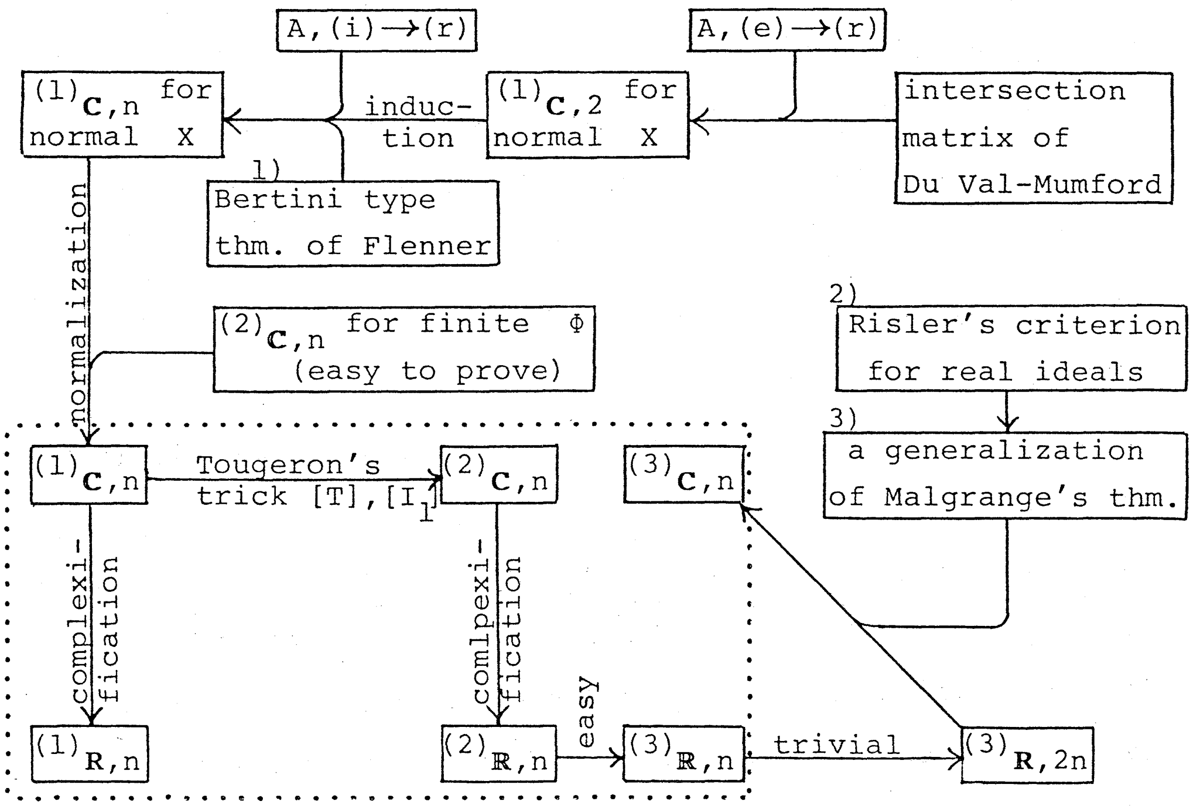
(2) If $\operatorname{grnk}_\eta \Phi = \dim X_\xi$, $\exists a_2 \geqq 1, \exists b_2 \geqq 0$: $\nu(f \circ \Phi) \leqq a_2 \nu(f) + b_2$ $(f \in O)$.

(3) If A is an open subanalytic set adherent to ξ, $\exists a_3 \geqq 1$,

$\exists b_3 \geqq 0$: $\mu_{A,\xi}(f) \leqq a_3 \nu(f)+b_3 \quad (f \in O)$.

If O is not an integral domain, a_1 does not exist (inf a_1 $=\infty$). So we may consider 1/inf a_1 as the distance of O from non-integral domains. If $X=k$, inf a_2 coincides with the reduced order of function Φ. Hence inf a_2 may be seen as a kind of order of analytic map Φ. 1/inf a_3 expresses something about the size of A_ξ.

If $n:=\dim X_\xi=1$, B,(1) follows from A,(e)$\rightarrow$(r) immediately. The rest is proved in the following way ($(1)_{\mathbf{C},n}$ means the assertion (1) for n-dimensional complex space, etc.).



1) Let (O,m) be an analytically irreducible local k-algebra and let $\varphi_1,\dots,\varphi_p \in m$. Flenner has given a sufficient

condition under which a general linear combination of φ_i generate a prime ideal of O (even of $\hat{O}$).

2) In the real case Nullstellensatz does not hold for radicals but for real ideals (real radicals). Risler has proved that, if $I \subset \mathbb{R}\{x\}$ is a prime ideal, I is a real ideal iff $\dim \mathbb{R}\{x\}/I =$ (topological dimension of $V(I)$ (the real analytic germ defined by I)).

3) A germ X_ξ of a complex analytic set can be canonically considered as a germ X_ξ^r of a real analytic set. Malgrange has proved that, if X_ξ is irreducible, X_ξ^r is also so together with its complexification $\tilde{X}_\xi$. We use the ringed space version of this theorem.

4. p-th power in $\mathbb{C}\{x\}$.

Let p be a prime number and suppose that u is not a p-th power in $\mathbb{C}\{x\}$ $(x=(x_1,\ldots,x_n))$. If we apply B,(1) to $O=\mathbb{C}\{x,y\}/(u-(-y)^p)$, we have the following.

C. Theorem. There exist $a \geqq p$, $b \geqq \nu(u)$, $b' \geqq 0$ depending only on n, p, u such that $a\nu(f)+b \geqq \nu(f^pu-g^p)$, $a\nu(g)+b' \geqq \nu(f^pu-g^p)$ $(f,\ g \in \hat{O})$.

We can consider $\theta(p,u):=p/\inf a$ as a distance of u from p-th powers in $\mathbb{C}\{x\}$.

5. Some problems and remarks.

1) Let X be a complex space reduced and irreducible everywhere. Is a_1 in B locally bounded? cf. [R_2],p.259.

2) Let $\Phi: Y \longrightarrow X$ be an analytic map such that X is reduced

and irreducible at $\xi=\Phi(\eta)$. Does existence of a_1 imply Gabrielov's regularity: $\mathrm{grnk}_\eta \Phi = \dim X_\xi$? cf. [B].

3) Do B, (1) and C hold for more general rings and ideals?

Example 3. Let A be a finitely generated ring over $\mathbb{C}$ and m one of its maximal ideal. Suppose that the completion $\hat{A}_m$ of the localization A_m is an integral domain. Then B, (1) implies a similar inequality for A.

4) Are $\inf a_1$ and $\inf b_1$ attained? Are they rational numbers?

5) If g is fixed, B, (1) holds with $a_1=1$ by the theorem of Artin-Rees. If we do not care the linearlity in the inequalities, B, (1) and C follows from Artin's strong approximation theorem ([A]).

(The detailed proofs of the results will be given in [I_2]).

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